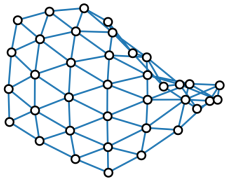
Graph Drawing and Tutte's Theorem

Ian Chen



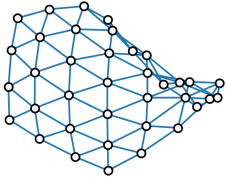
Graphs



Spring



Graphs



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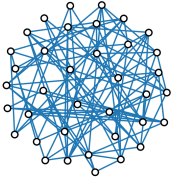
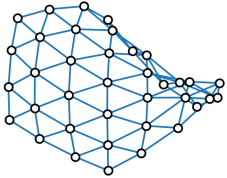


ARF



Graphs



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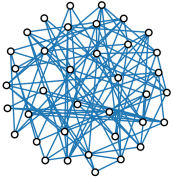

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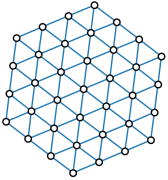
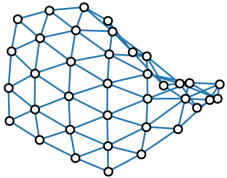


Kamada–Kawai



Graphs



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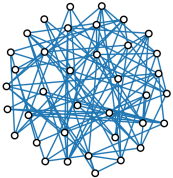

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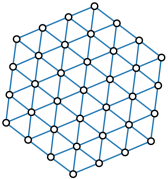

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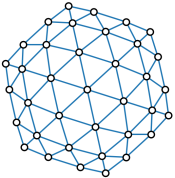
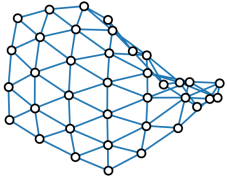


Spectral



Graphs



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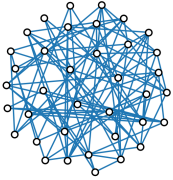

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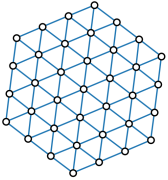

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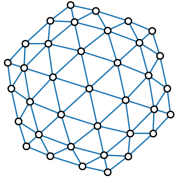

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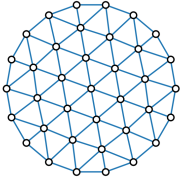

Tutte



Outline

Overview
- Tutte
- Other force-directed

Tutte's Theorem



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Tutte's Theorem



Tutte's Spring Embedding

- Graph $G = (V, E)$, embedding $p : V \to \mathcal{R}^2$ (denote $\boldsymbol{p}_v = p(v)$).



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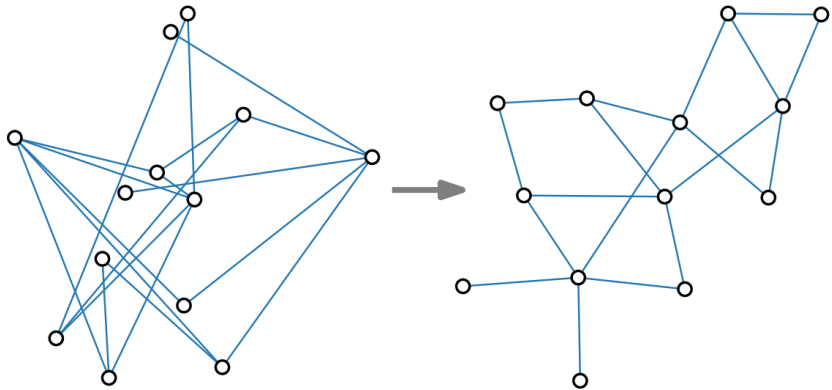



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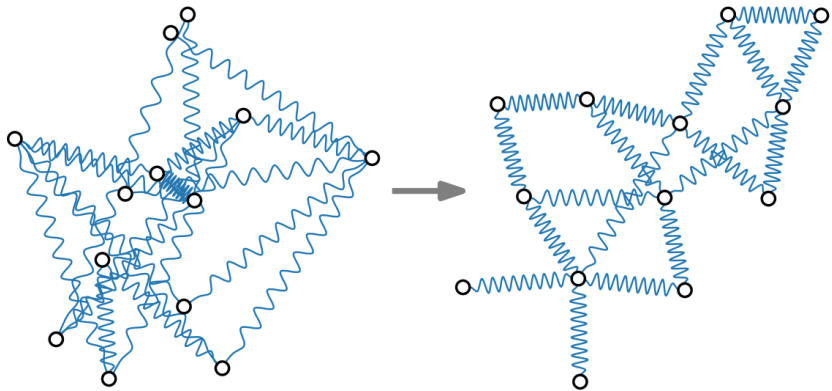
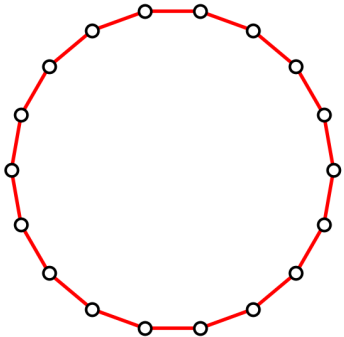




Tutte's Spring Embedding (cont.)

- boundary S positioned at convex polygon; free vertices $V \setminus S$





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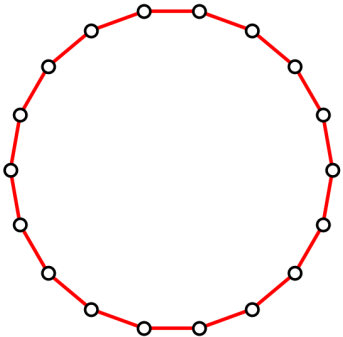

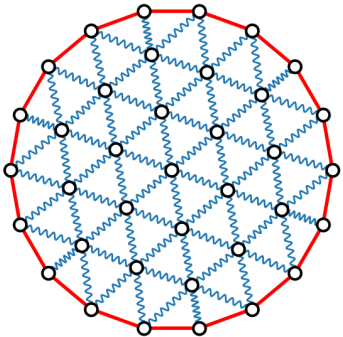



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Tutte's Spring Embedding (cont.)

- Theorem: (Whitney '33) every 3-connected planar graph has a unique embedding[1]



[1] up to homeomorphism, on the sphere

Tutte's Spring Embedding (cont.)

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Kamada-Kawai layout

- spring equilibrium based on graph distances d_{uv}



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Anti-gravity drawing

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Tutte's Spring Embedding (cont.)

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  - $\Longrightarrow$ planar embedding



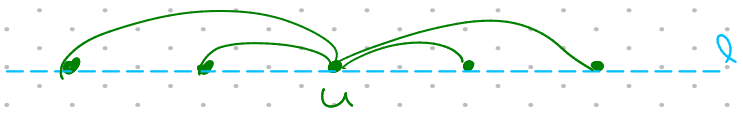
No degenerate Neighborhoods

Equilibrium condition:

$$\sum_v (p_v - p_u) = 0$$



No degenerate Neighborhoods



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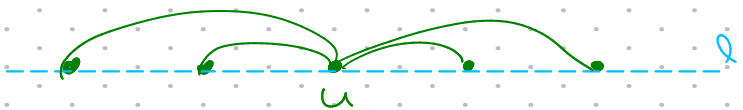

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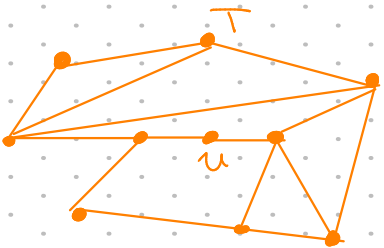

Tutte Embedding of G.

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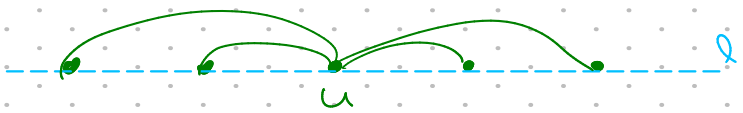

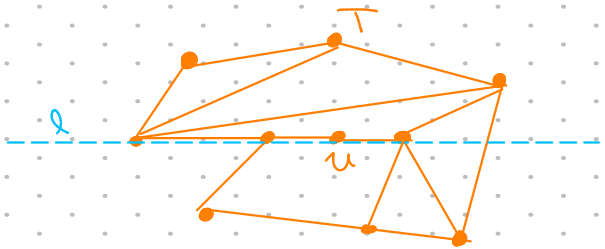

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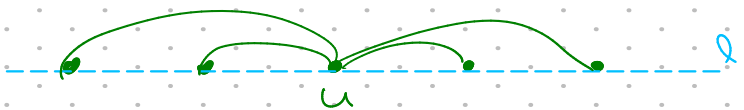

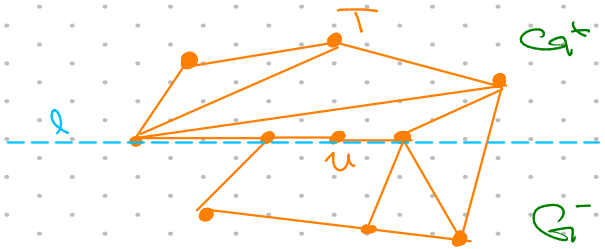

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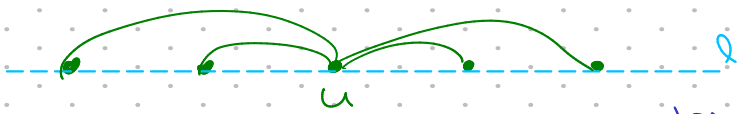

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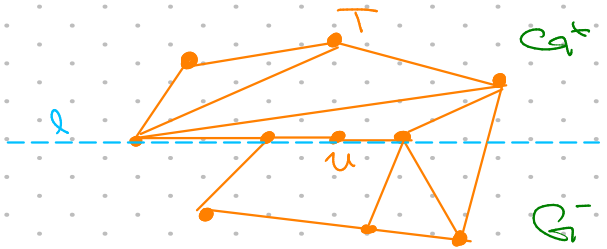

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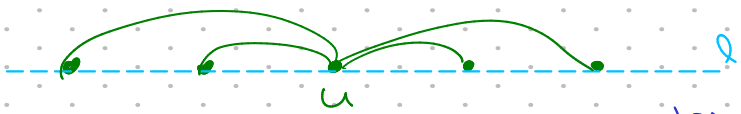

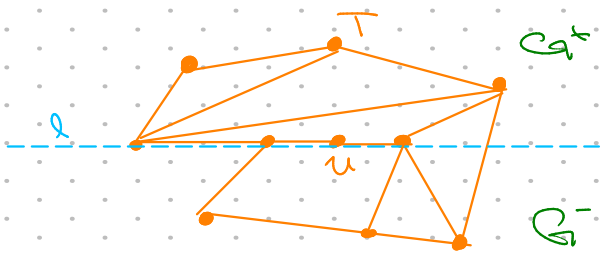

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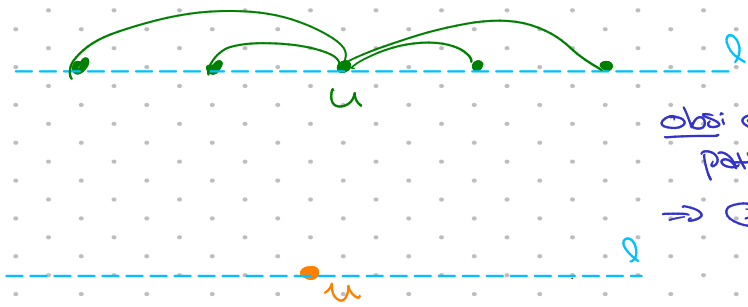

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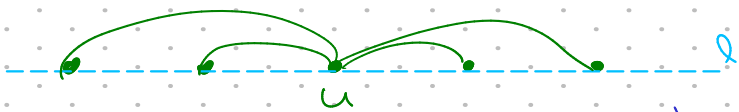

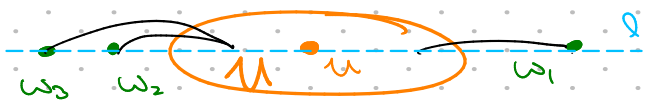

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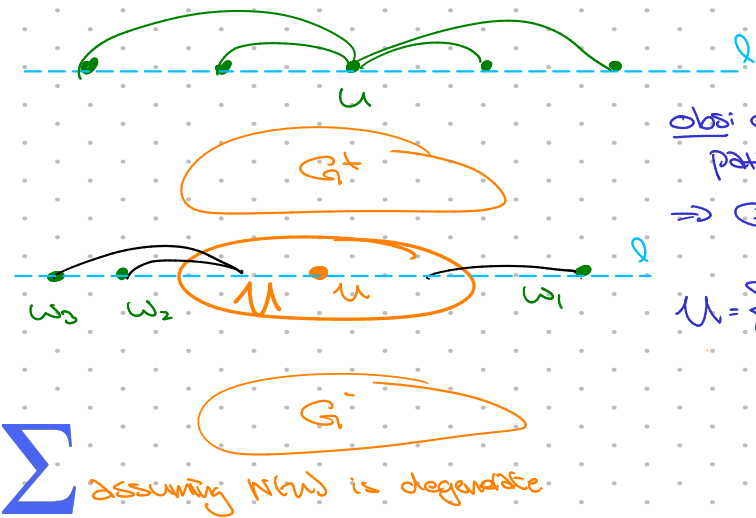

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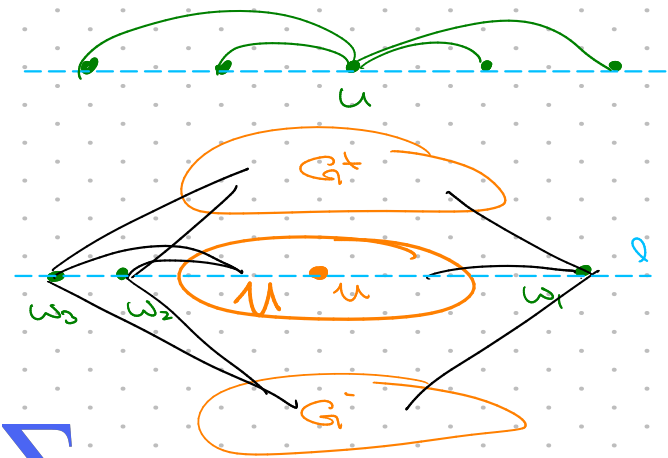

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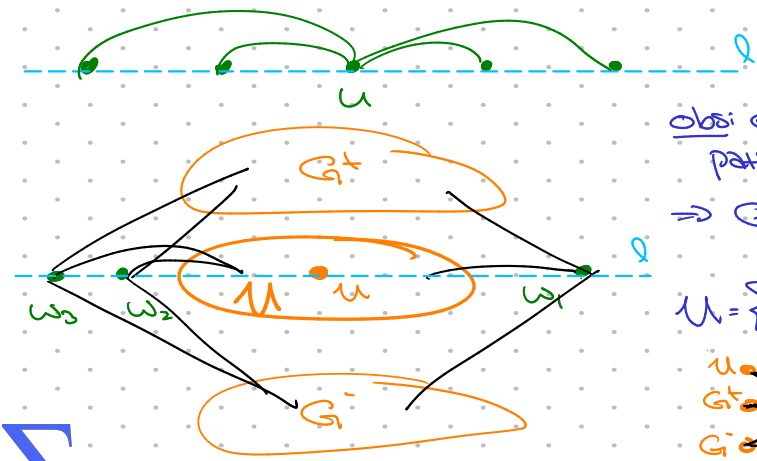

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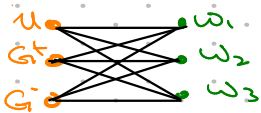
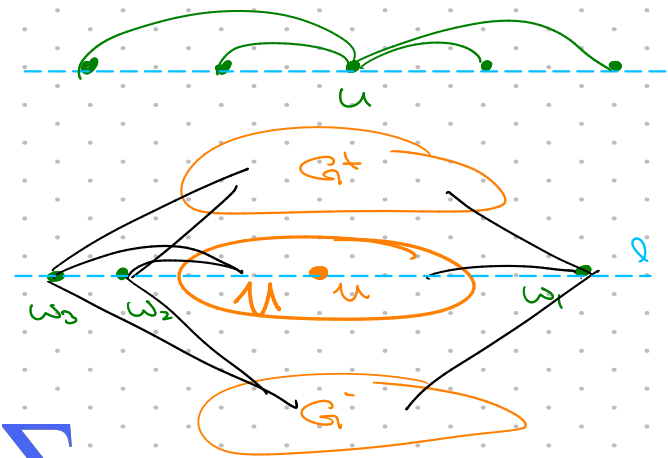


$\sum$ assuming $N(u)$ is degenerate

No degenerate Neighborhoods



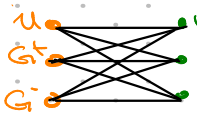
Equilibrium condition:

$$\sum_v (p_v - p_u) = 0$$

Obs: every vertex has nondecreasing path to T (by height)

$\Rightarrow G^+, G^-$ connected

$U = \{ v \in V : N(v) \text{ on } \ell \}$



$K_{3,3}$ minor $\Rightarrow$ not planar ⚠

$\sum$ assuming $N(u)$ is degenerate

Proof outline

- No degenerate neighborhoods (not all neighborhood lies on a line)



Proof outline

- No degenerate neighborhoods (not all neighborhood lies on a line)
  - $\implies$ every free vertex has neighbors strictly on both sides of any line



Proof outline

- No degenerate neighborhoods (not all neighborhood lies on a line)
  - $\Longrightarrow$ every free vertex has neighbors strictly on both sides of any line
  - $\Longrightarrow$ all free vertices inside the outer polygon



Split-Faces Lemma:

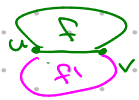


Equilibrium Condition:

$$\sum_v (p_v - p_u) = 0$$

Σ

Split-Faces Lemma:



Equilibrium Condition:

$$\sum_v (p_v - p_u) = 0$$

Σ

Split-Faces Lemma:



f and f' are on (strictly) separate sides of ℓ

Equilibrium Condition:

$\sum_v (p_v - p_u) = 0$



Split-Faces Lemma:



f and f' are on (strictly) separate sides of l

Equilibrium Condition:

$\sum_v (p_v - p_u) = 0$



$\sum$ position vertices to any plane always

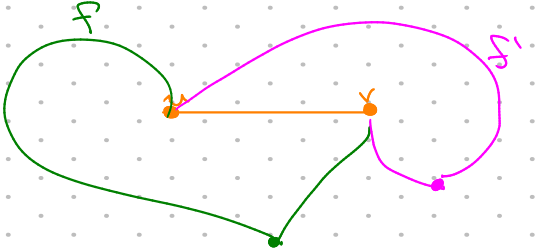
Split-Faces Lemma:



f and f' are on (strictly) separate sides of ℓ

Equilibrium Condition:

$$\sum_v (p_v - p_u) = 0$$



$\sum$ position vertices to any planar drawing

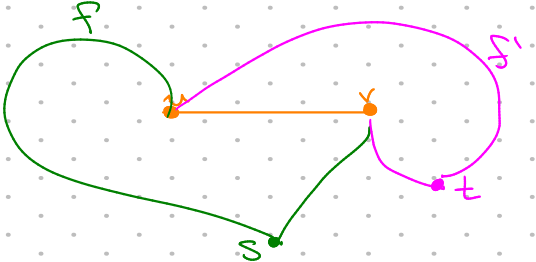
Split-Faces Lemma:



f and f' are on (strictly) separate sides of l

Equilibrium Condition:

$\sum_v (p_v - p_u) = 0$



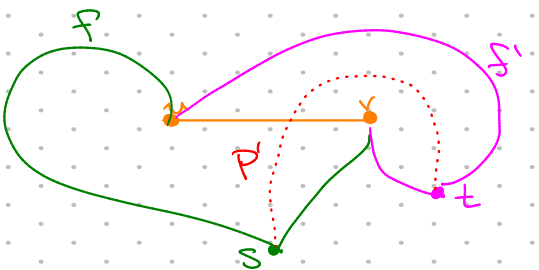
$\sum$ position vertices to any plane drawing

Split-Faces Lemma:



f and f' are on (strictly) separate sides of l

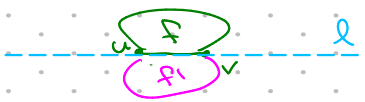
Equilibrium Condition:
$\sum_{v}(p_v - p_u) = 0$



① P' curve in $\mathbb{R}^2$ from s-t contained in f, f'

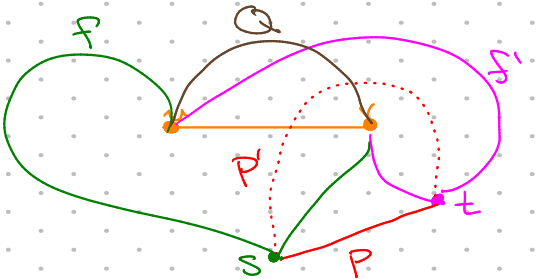
Σ position vertices to any point displacing

Split-Faces Lemma:



f and f' are on (strictly) separate sides of l

Equilibrium Condition:
$\sum_v (p_v - p_u) = 0$



① P' curve in $\mathbb{R}^2$ from s-t contained in f, f'

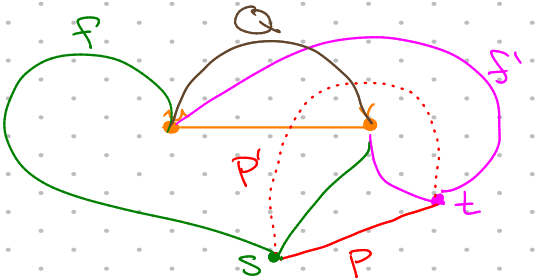
Σ position vertices to any place allowing

Split-Faces Lemma:



f and f' are on (strictly) separate sides of l

Equilibrium Condition:
$\sum_v (p_v - p_u) = 0$

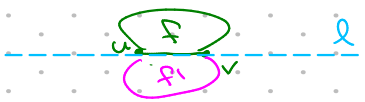


(1) P' curve in $\mathbb{R}^2$ from s-t contained in f, f'

(2) Any Q (u→v) must intersect P (s→t)

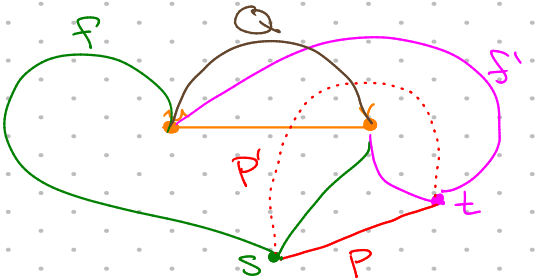
$\sum$ position vertices to any point displacing

Split-Faces Lemma:



f and f' are on (strictly) separate sides of l

Equilibrium Condition:
$\sum_v (P_v - P_u) = 0$



① P' curve in $\mathbb{R}^2$ from s-t contained in f, f'

② Any Q (u→v) must intersect P (s→t)

③ If s, t on same side:
⇒ choose P in V^-
⇒ choose Q in V^+

∎

$\sum$ position vertices to any point displacing

Proof outline

- Split-faces lemma (adjacent faces strictly separated by a line)



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  - $\Longrightarrow$ strictly-convex faces



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  - $\Longrightarrow$ no degenerate vertices (embedding injective)



Proof outline

- Split-faces lemma (adjacent faces strictly separated by a line)
  - $\Longrightarrow$ strictly-convex faces
  - $\Longrightarrow$ no degenerate vertices (embedding injective)
- Points are in exactly one face



Proof outline

- Split-faces lemma (adjacent faces strictly separated by a line)
  - $\Longrightarrow$ strictly-convex faces
  - $\Longrightarrow$ no degenerate vertices (embedding injective)
- Points are in exactly one face
  - $\Longrightarrow$ planar embedding

