What's the big deal with Expanders (Part 1)

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Expanders...

have...

- no small cuts
- small mixing times
- evenly distributed edges
- many vertex disjoint paths

and are used to construct...

- pseudorandom number generators
- routing networks
- constrant rate codes
- error amplifiers

Preliminaries

- Combinatorial expanders G have $|N(S)| \geq \beta|S|$ for all $S \subset V$
- Spectral expander (families) have $\lambda_2[\frac{1}{2}(I + AD^{-1})]$ bounded away from 0
- By Cheeger's inequality, they are essentially the same (up to constants)

Theorem (Noga Alon, Theorem 1.3, 2020)

For every even $d > 2$, $\exists n_0(d)$ for explicit construction of G_n d-regular expander family, $n \geq n_0$.

Alon, Noga. "Explicit expanders of every degree and size." Combinatorica 41.4 (2021): 447-463.

Preliminaries (cont.)

- A left d-regular bipartite graph $G = (L \cup R, E)$ is an (α, β) expander if

$$|N(S)| \geq \beta|S| \qquad \forall S \subset V, |S| \leq \alpha|L|$$

- This week, we show applications of bipartite combinatorial expanders:
  - construct routing networks
  - construct constant rate codes
- Next week, spectral expanders:
  - Random walks
  - PRNGs
  - PCP Theorem

Routing Networks

- Given n input and output terminals, build a network G s.t.
  - Any permutation can be routed
  - For any $f : [\![n]\!] \to [\![n]\!]$, exists vertex disjoint paths $\pi_i : i \mapsto f(i)$
  - G has bounded degree, $\tilde{O}(n)$ vertices
  - Routes can be efficiently found (distributed)
- This is known as non-blocking flows

Routing Networks (cont.)

- Butterfly networks are small bounded degree and connected:
  - Each of the $\log n$ layers has n vertices
  - Routes are found by flipping the bits one at a time
  - Edges between layer ℓ and layer $\ell + 1$:

$$(v, \ell) \to (v, \ell + 1), \qquad (v, \ell) \to (v + 2^{\ell}, \ell + 1)$$

- More abstractly, layer ℓ has 2^{ℓ} blocks of size 2^{ℓ}
  - Layer ℓ connected to 2 blocks in layer $\ell + 1$
  - Layer $\ell + 1$ connected to 2 blocks in layer ℓ
- But they do not support non-blocking flows (the unique paths intersect)

Routing Networks (cont.)

- d-multibutterfly networks have connections being two blocks as a d-regular (α, β) bipartite expander, $\beta > 1$
- For subsets S, $|S| \leq \alpha n$, find matching between S and $N(S)$
  - Inducing on each block forms an (α, β) expander
  - Halls' theorem guarantee existence of a matching between each layer
- Efficient distributed routing algorithms exist
  - In layer $\ell + 1$, match if unique waiting vertex in layer ℓ
  - Guarantee (poly)-logarithmic number of rounds

A Constant Rate Code

- Communicating 2^n symbols across a noisy channel
- Encode as n/r length bit strings (rate r)
- Redundancy allows us to be robust to errors
- For *gap* k, decode correctly if $\leq k$ bitflips in the noisy channel
- Expander codes $\implies$ constant rate, linear gap, linear time decoding alg
  - Using d-regular $(\alpha, \frac{3d}{4})$ expanders:
  - Rate $r = 1/2$, gap $k = \alpha n$

A Constant Rate Code (cont.)

- $G = (L \cup R, E)$, where $|L| = 2n$, $|R| = n$ expander
- Each of $2n$ bits indexed by some $u \in L$
- Valid codewords have $\sum_{u \in N(v)} b_u \equiv_2 0$, for $v \in R$
- Any two valid codewords have at differ in many places:
  - If $S \subset L$ are the different bits, $|S| \leq 2\alpha n$ contradiction:

$$d|S| = E(N(S), S) \geq 2|N(S)| > 2\frac{3d}{4}|S|$$

  - Tighter bounds gets $|S| \geq 4\alpha n$

A Constant Rate Code (cont.)

- Decoding algorithm by:
  - Find $u \in L$ with $> d/2$ unsatisfied neighbors
  - Flip b_u and repeat
- Each time we reduce number of unsatisfied $v \in R$, so linear steps
- Given k distance to nearest codeword, ℓ number of satisfied clauses (in R),

$$d|S| = E(S, N(S)) > 2\ell + (3d/4|S| - \ell) = \frac{3d}{4}|S| + \ell$$

- Thus, number of unsatisfied clauses $> \frac{d}{2}k$, so always converge to codeword
- Starting at $\leq \alpha n$ distance, then stay $\leq 2\alpha n$ distance (ℓ strictly increasing)