Y_t = # of distinct rolls after t throws.

T_k = # of throws until hit k distinct rolls.

<u>Obs:</u> $\underbrace{\text{Prob}(Y_t \geq k)}_{\text{getting at least } k \text{ distinct in } t \text{ throws}} = \underbrace{\text{Prob}(T_k \leq t)}_{\text{having } k^{th} \text{ unique arrive before } t^{th} \text{ throw}}$

Thus, $\text{Prob}(Y_6 = 4) = \text{Prob}(Y_6 \geq 4) - \text{Prob}(Y_6 \geq 5)$

$= \underbrace{\text{Prob}(T_4 \leq 6) - \text{Prob}(T_5 \leq 6)}$ (*)

For the calculation let $p_i = \frac{7-i}{6}$, $G_i = \text{Geometric}(p_i)$, $i = 1, 2, \ldots, 5$

so that $T_1 \stackrel{D}{=} G_1$ $T_2 \stackrel{D}{=} G_1 + G_2$, $\ldots$, $T_k = \sum_{i=1}^{k} G_i$

To compute the probability density function of T_k, we use probability generating functions, P:

$$P_X(t) = \mathbb{E}[t^X] = \sum_{x=1}^{\infty} Pr(X=x)\, t^x$$

and $P_{Geom(p)}(t) = \sum_{x=1}^{\infty} Pr(Geom(p)=x)\, t^x = pt \sum_{x=0}^{\infty} (1-p)^x t^x$

$= \left[pt / 1-(1-p)t \right]$

so that

(independence $\downarrow$ $G_i \perp\!\!\!\perp G_j,\ i \neq j$)

$$P_{T_4}(t) = \mathbb{E}\left[t^{(G_1+G_2+G_3+G_4)} \right] = P_{G_1}(t) \times P_{G_2}(t) \times P_{G_3}(t) \times P_{G_4}(t)$$

$$= \overbrace{\left(1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6}\right) t^4}^{\text{numerators}} \times \overbrace{\left(\frac{1}{1-t/6} \times \frac{1}{1-t/3} \times \frac{1}{1-t/2}\right)}^{\text{denom.}} = \left(\frac{5t^4}{18}\right) \underbrace{\left(\frac{A}{1-t/6} + \frac{B}{1-t/3} + \frac{C}{1-t/2}\right)}$$

$\uparrow$ partial frac

To find A, take $A = \lim_{t \to 6} \left[\frac{A}{1-t/6} + \frac{B}{1-t/3} + \frac{C}{1-t/2} \right] \times \left[1 - \frac{t}{6} \right]$

$$= \lim_{t\to 6}\left[\frac{1}{1-t/6}\times\frac{1}{1-t/3}+\frac{1}{1-t/2}\right]\times\left[1-t/6\right] = 1\times\frac{1}{1-2}\times\frac{1}{1-3} = \frac{1}{2}$$

To find B, take $B = \lim_{t\to 3}\left[\frac{A}{1-t/6}+\frac{B}{1-t/3}+\frac{C}{1-t/2}\right]\times\left[1-t/3\right]$

$$= \lim_{t\to 3}\left[\frac{1}{1-t/6}+\frac{1}{1-t/3}+\frac{1}{1-t/2}\right]\times\left[1-t/3\right] = \frac{1}{1-1/2}\times 1\times\frac{1}{1-3/2} = -4$$

To find C, take $C = \lim_{t\to 2}\left[\frac{A}{1-t/6}+\frac{B}{1-t/3}+\frac{C}{1-t/2}\right]\times\left[1-t/2\right]$

$$= \lim_{t\to 2}\left[\frac{1}{1-t/6}+\frac{1}{1-t/3}+\frac{1}{1-t/2}\right]\times\left[1-t/2\right] = \frac{1}{1-1/3}\times\frac{1}{1-2/3}\times 1 = \frac{9}{2}$$

Thus, $\mathbb{E}\left[t^{T_4}\right] = \left(\frac{5t^4}{18}\right)\left[2\,\frac{1}{1-t/6} - 4\,\frac{1}{1-t/3} + \frac{9}{2}\,\frac{1}{1-t/2}\right]$

$$= \left(\frac{5t^4}{18}\right)\left[2\sum_{k=0}^{\infty}\left(\frac{t}{6}\right)^k - 4\sum_{k=0}^{\infty}\left(\frac{t}{3}\right)^k + \frac{9}{2}\sum_{k=0}^{\infty}\left(\frac{t}{2}\right)^k\right]$$

and $\text{Prob}(T_4 \le 6) = \left(\frac{5}{18}\right)\left[2\left(1+\frac{1}{6}+\frac{1}{36}\right) - 4\left(1+\frac{1}{3}+\frac{1}{9}\right) + \frac{9}{2}\left(1+\frac{1}{2}+\frac{1}{4}\right)\right]$

(1)

Finally, we repeat this calculation for T_5:

$$\mathbb{E}\left(t^{T_5}\right) = t^5\left(1\times\frac{5}{6}\times\frac{4}{6}\times\frac{3}{6}\times\frac{2}{6}\right)\times\frac{1}{1-t/6}\times\frac{1}{1-t/3}\times\frac{1}{1-t/2}\times\frac{1}{1-2t/3}$$

$$= \left(t^5\right)\left(\frac{5}{54}\right)\left[\frac{A}{1-t/6}+\frac{B}{1-t/3}+\frac{C}{1-t/2}+\frac{D}{1-2t/3}\right]$$

Where $A = \lim_{t\to 6}\left(\left(1-\frac{t}{6}\right)\times\left(\frac{A}{1-t/6}+\frac{B}{1-t/3}+\frac{C}{1-t/2}+\frac{D}{1-2t/3}\right)\right)$

$$= \lim_{t\to 6}\left(1-\frac{t}{6}\right)\left(\frac{1}{1-t/6}\times\frac{1}{1-t/3}\times\frac{1}{1-t/2}\times\frac{1}{1-2t/3}\right) = -\frac{1}{6}$$

$$B = \lim_{t\to 3}\left(1-\frac{t}{3}\right)\left(\frac{1}{1-t/6}\times\frac{1}{1-t/3}\times\frac{1}{1-t/2}\times\frac{1}{1-2t/3}\right) = 4$$

$$C = \lim_{t\to 2}\left(1-\frac{t}{2}\right)\left(\frac{1}{1-t/6}\times\frac{1}{1-t/3}\times\frac{1}{1-t/2}\times\frac{1}{1-2t/3}\right) = -\frac{27}{2}$$

$$D = \lim_{t \to 3/2} \left(1 - \frac{2t}{3}\right)\left(\frac{1}{1-t/6} \times \frac{1}{1-t/3} \times \frac{1}{1-t/2} \times \frac{1}{1-2t/3}\right) = \frac{34}{3}$$

Thus, $$\mathbb{E}\left[t^{T_5}\right] = \left(\frac{5t^5}{54}\right) \sum_{k=0}^{\infty} \left[-\frac{1}{6}\left(\frac{t}{6}\right)^k + 4\left(\frac{t}{3}\right)^k - \frac{27}{2}\left(\frac{t}{2}\right)^k + \frac{34}{3}\left(\frac{2t}{3}\right)^k\right]$$

So $$\mathrm{Prob}(T_5 \le 6) \overset{(2)}{=} \frac{5}{54}\left(-\frac{1}{6}\left(1+\frac{1}{6}\right) + 4\left(1+\frac{1}{3}\right) - \frac{27}{2}\left(1+\frac{1}{2}\right) + \frac{34}{3}\left(1+\frac{2}{3}\right)\right)$$

Finally conclude from (1) - (2) $\approx \boxed{0.515}$